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Delay-independent stability criteria for a class of retarded dynamical systems with two delays

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Abstract

In this paper, sufficient and necessary delay-independent stability criteria are given for a class of retarded dynamical systems with two discrete time delays and parameters. The delay-independent stability problem of the system is discussed in terms of the stability of the characteristic function, which is determined by checking the existence of real roots for some polynomials. The criteria are also illustrated by examples.

1. Introduction

In this paper the delay-independent stability, i.e., the asymptotic stability independent of delays [1], is discussed for the linear retarded system

$$\dot{x}(t) = A_0x(t) + A_1x(t-\tau_1) + A_2x(t-\tau_2), \tag{1}$$

where $t\in[0,+\infty]\triangleq\bar{\mathbb{R}}^+$; A_0, A_1, $A_2\in\mathbb{R}^{n\times n}$, $\mathbb{R}=(-\infty,+\infty)$, $n\geqslant 1$; $\tau_1,\tau_2\in\bar{\mathbb{R}}^+$ are time delays; $x(t)$, $x(t-\tau_1)$, $x(t-\tau_2)\in\mathbb{R}^{n\times 1}$; $rank(A_1)=rank(A_2)=1$.

System (1) may result from the linearization of a non-linear system around its equilibrium. Consider the non-linear control system

$$\dot{x}_s(t) = f[x_s(t), u_s(t)],$$
$$y_s(t) = g[x_s(t)], \tag{2}$$

where $t\in\bar{\mathbb{R}}^+$; $x_s(t)\in\mathbb{R}^{n\times 1}$, $u_s(t)\in\mathbb{R}^{2\times 1}$ and $y_s(t)\in\mathbb{R}^{l\times 1}$ are the state, control and output vectors, respectively, $l\geqslant 1$; $f:\mathbb{R}^{n\times 1}\times\mathbb{R}^{2\times 1}\to\mathbb{R}^{n\times 1}$, $g:\mathbb{R}^{n\times 1}\to\mathbb{R}^{l\times 1}$ and are such that solutions to initial

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value problems exist and are continuable. The linearized system around an equilibrium of (2) can be written as

$$\dot{x}(t) = Ax(t) + Bu(t),$$

$$y(t) = Cx(t),$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 2}$, $C \in \mathbb{R}^{l\times n}$; $x(t) \in \mathbb{R}^{n\times 1}$, $u(t) \in \mathbb{R}^{2\times 1}$ and $y(t) \in \mathbb{R}^{l\times 1}$ are the state, control and output vectors around the equilibrium, respectively. When $u(t) = [u_1(t)\ u_2(t)]^{\mathrm{T}}$ is governed by the static delayed output feedback [2] in the form

$$u_1(t) = \gamma_1^{\mathrm{T}} y(t - \tau_1),$$
$$u_2(t) = \gamma_2^{\mathrm{T}} y(t - \tau_2),$$

where γ_1^{T}, $\gamma_2^{\mathrm{T}} \in \mathbb{R}^{1\times l}$ are feedback gain matrices, the linearized system is in form of Eq. (1) with

$$A_0 = A, \quad A_1 = \beta_1 \gamma_1^{\mathrm{T}} C, \quad A_2 = \beta_2 \gamma_2^{\mathrm{T}} C,$$

where $B = [\beta_1\ \beta_2]$ and β_1, $\beta_2 \in \mathbb{R}^{n\times 1}$.

The stability for linear retarded systems has caught the attention of researchers for a long time [3–6]. Other than being solved with the Liapunov method [5,2], this stability problem is usually transformed into that of the corresponding characteristic function [4,5,7], for which two kinds of stability are studied: the delay-dependent stability and the delay-independent stability [2].

There are numerous literatures on the delay-independent stabilities. Results that are most closely related to this paper are [1,8–11]. Hale et al. [10] developed a general analytic criterion for the delay-independent stability of the system (and generalized system (2.1) in [10])

$$\dot{x}(t) = A_0 x(t) + \sum_{k=1}^{N} A_k x(t - \tau_k), \tag{3}$$

where $x(t) \in \mathbb{R}^{n\times 1}$, $A_k \in \mathbb{R}^{n\times n}$, $\tau_k \in \bar{\mathbb{R}}^{+}$, $k = 1, 2, \ldots, N$. But to apply this criterion, much work has to be done even for the case with $N = 1$, $n = 2$, for which Chin [8] independently gave a complex algebraic discrimination method at an earlier time. Chen and Latchman [1] provided the method of frequency sweeping tests to determine the delay-independent stability of (3) when (1) $\tau_1, \tau_2, \ldots, \tau_N$ are independent, or (2) $\tau_k = k\tau$, $k = 1, 2, \ldots, N$. For case (2), Niculescu [11] also gave a discrimination method based on the matrix pencil technology. Generally, the methods in [1,11] both demand there are no parameters in systems, i.e., each A_k is known. As for the case when there exist parameters in systems, an approach was contributed by Wang and Hu [9] in studying the delay-independent stability for systems with the characteristic function

$$D(\lambda, \tau_1, \tau_2) = P_1(\lambda)\mathrm{e}^{-\lambda\tau_1} + P_2(\lambda)\mathrm{e}^{-\lambda\tau_2} + P_0(\lambda), \tag{4}$$

where $P_1(\lambda)$, $P_2(\lambda)$ and $P_0(\lambda)$ are polynomials, the coefficients of which are all real-valued functions of the system parameters; the leading coefficient of $P_0(\lambda)$ is assumed to be 1; $\deg[P_0(\lambda)] > \max\{\deg[P_1(\lambda)], \deg[P_2(\lambda)]\}$. The authors transformed the delay-independent stability problem into that of the non-existence of the real roots for polynomials, and got the delay-independent stability region in the space of parameters by the generalized Sturm theory [12,13].

However, all the literatures [1,8,9,11] do not cover the delay-independent stability of system (1) with parameters, for which the characteristic function is shown in Section 2 to be

$$D(\lambda,\tau_1,\tau_2) = P_{12}(\lambda)\mathrm{e}^{-\lambda(\tau_1+\tau_2)} + P_1(\lambda)\mathrm{e}^{-\lambda\tau_1} + P_2(\lambda)\mathrm{e}^{-\lambda\tau_2} + P_0(\lambda), \tag{5}$$

where the symbols and conditions are similar to those in Eq. (4) in addition that $P_{12}(\lambda)$ is a real coefficient polynomial and $\deg[P_{12}(\lambda)] < \min\{\deg[P_1(\lambda)], \deg[P_2(\lambda)]\}$.

Based on the pioneering works [1,8–11], this paper studies the delay-independent stability of Eq. (1) with parameters in cases of two independent delays (τ_1 and τ_2 are two independent variables) and two dependent delays ($\tau_1 = h_1\tau$, $\tau_2 = h_2\tau$, where h_1 and h_2 are positive integers; $\tau \geqslant 0$).

The organization of this paper is as follows. In the next section, the characteristic function is derived. In Sections 3 and 4, the delay-independent stability criteria are developed for Eq. (1), and are illustrated with examples including a retarded stirred tank system. The conclusion follows in the final section.

2. Derivation of the characteristic function

The characteristic function of Eq. (1) is shown to be Eq. (5). Assuming that $n \geqslant 2$ in system (1) (Note that the case for $n = 1$ is obvious). Because $rank(A_1) = rank(A_2) = 1$, there exists non-zero matrices $\beta_1, \beta_2 \in \mathbb{R}^{n\times 1}$, $\kappa_1^{\mathrm{T}}, \kappa_2^{\mathrm{T}} \in \mathbb{R}^{1\times n}$ such that $A_1 = \beta_1\kappa_1^{\mathrm{T}}$, $A_2 = \beta_2\kappa_2^{\mathrm{T}}$, where

$$\beta_1 = [b_{11} \quad b_{21} \quad \cdots \quad b_{n1}]^{\mathrm{T}}, \quad \beta_2 = [b_{12} \quad b_{22} \quad \cdots \quad b_{n2}]^{\mathrm{T}},$$
$$\kappa_1^{\mathrm{T}} = [k_{11} \quad k_{12} \quad \cdots \quad k_{1n}], \quad \kappa_2^{\mathrm{T}} = [k_{21} \quad k_{22} \quad \cdots \quad k_{2n}].$$

Let $K = \left[\begin{smallmatrix}\kappa_1^{\mathrm{T}}\\ \kappa_2^{\mathrm{T}}\end{smallmatrix}\right] = [K_1 \;\; K_2]$, where $K_1 \in \mathbb{R}^{(n-2)\times 2}$ and $K_2 \in \mathbb{R}^{2\times 2}$ (K_1 is omitted when $n = 2$; the rest may be deduced by analogy). Let $B = [\beta_1 \;\; \beta_2]$, and consider two cases:

(1) β_1 and β_2 are linearly independent. There exists a partition of B, such that $B = \left[\begin{smallmatrix}B_1\\ B_2\end{smallmatrix}\right]$, where $B_1 \in \mathbb{R}^{(n-2)\times 2}$, $B_2 \in \mathbb{R}^{2\times 2}$ and B_2 is non-singular (linear transformation to Eq. (1) can be performed when necessary). Choose two non-singular real matrices B_Δ and P, where (note that $I \in \mathbb{R}^{(n-2)\times(n-2)}$ is the unit matrix)

$$B_\Delta = diag\{b_{11}, b_{22}\}, \quad P = \begin{bmatrix} I & -B_1B_2^{-1} \\ 0 & B_\Delta B_2^{-1} \end{bmatrix},$$

such that

$$P[\beta_1 \quad \beta_2] = PB = \begin{bmatrix} I & -B_1B_2^{-1} \\ 0 & B_\Delta B_2^{-1} \end{bmatrix}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = [b_{\Delta 1} \quad b_{\Delta 2}],$$

where

$$b_{\Delta i} = \underbrace{[\,0 \;\; \cdots \;\; 0 \;\; b_{ii} \;\; 0 \;\; \cdots \;\; 0\,]^{\mathrm{T}}}_{b_{ii} \text{ is in the } (n-2+i) \text{ place}}, \quad i = 1, 2.$$

Thus

$$P\left(\sum_{i=1}^{2} \beta_i\kappa_i^{\mathrm{T}}\mathrm{e}^{-\lambda\tau_i}\right)P^{-1} = \sum_{i=1}^{2} b_{\Delta_i}\kappa_i^{\mathrm{T}}\mathrm{e}^{-\lambda\tau_i}\cdot\begin{bmatrix} I & -B_1B_\Delta^{-1} \\ 0 & B_2B_\Delta^{-1} \end{bmatrix} = \begin{bmatrix} 0 \\ E_\Delta B_\Delta B_K \end{bmatrix},$$

where

$$E_\Delta = diag[\mathrm{e}^{-\lambda\tau_1}, \mathrm{e}^{-\lambda\tau_2}], \quad B_K = [K_1(K_1B_1 + K_2B_2)B_\Delta^{-1}].$$

So the characteristic function of Eq. (1) is

$$\begin{aligned} D(\lambda,\tau_1,\tau_2) &= \det\left[\lambda I - A_0 - \left(\sum_{i=1}^{2} \beta_i \kappa_i^{\mathrm{T}} \mathrm{e}^{-\lambda\tau_i}\right)\right] = \det(\lambda I - A_P - B_P) \\ &= P_{12}(\lambda)\mathrm{e}^{-\lambda(\tau_1+\tau_2)} + P_1(\lambda)\mathrm{e}^{-\lambda\tau_1} + P_2(\lambda)\mathrm{e}^{-\lambda\tau_2} + P_0(\lambda), \end{aligned}$$

where

$$A_P = PA_0P^{-1}, \quad B_P = P\left(\sum_{i=1}^{2} \beta_i \kappa_i^{\mathrm{T}} \mathrm{e}^{-\lambda\tau_i}\right)P^{-1}. \tag{6}$$

(2) β_1 and β_2 are linearly dependent. Just as well suppose $b_{n1} \neq 0$. Then an $n \times n$ non-singular real matrix P can be found such that $P\beta_1 = [0\ 0\ \cdots\ 0\ b_{n1}]^{\mathrm{T}}$ and $P\beta_2 = [0\ 0\ \cdots\ 0\ b_{n2}]^{\mathrm{T}}$. So the characteristic function of Eq. (1) is

$$\begin{aligned} D(\lambda,\tau_1,\tau_2) &= \det\left[\lambda I - A_0 - \left(\sum_{i=1}^{2} \beta_i \kappa_i^{\mathrm{T}} \mathrm{e}^{-\lambda\tau_i}\right)\right] = \det[\lambda I - A_P - B_P] \\ &= P_1(\lambda)\mathrm{e}^{-\lambda\tau_1} + P_2(\lambda)\mathrm{e}^{-\lambda\tau_2} + P_0(\lambda), \end{aligned}$$

where A_P, B_P are defined the same way as those in Eq. (6).

3. Delay-independent stability analysis

Two stability analyses are carried out: one for two independent delays, and the other for two dependent delays. Theorems 3.1 and 3.2 are the corresponding two main criteria.

3.1. The case for two independent delays

Consider the case when τ_1 and τ_2 are independent.

Lemma 3.1. *The linear retarded dynamical system with characteristic function* (5) *is delay-independently stable if and only if*: (i) *The function* $D(\lambda, 0, 0) = P_{12}(\lambda) + P_1(\lambda) + P_2(\lambda) + P_0(\lambda)$ *is Hurwitz stable*, *and* (ii) *equation* $D(\lambda, \tau_1, \tau_2) = 0$ *has no non-zero root* λ *on the imaginary axis for any given delays* τ_1 *and* τ_2.

Proof. This lemma can be proved by applying Theorem 2.4 in Ref. [10] to (1). □

Let

$$\begin{aligned} &P_{12}(\mathrm{i}\omega) = P_{12R}(\omega) + \mathrm{i}P_{12I}(\omega), \quad P_1(\mathrm{i}\omega) = P_{1R}(\omega) + \mathrm{i}P_{1I}(\omega), \quad \mathrm{i} = \sqrt{-1}, \\ &P_2(\mathrm{i}\omega) = P_{2R}(\omega) + \mathrm{i}P_{2I}(\omega), \quad P_0(\mathrm{i}\omega) = P_{0R}(\omega) + \mathrm{i}P_{0I}(\omega), \quad \omega \in \mathbb{R}, \end{aligned}$$

where $P_{12R}(\omega)$, $P_{12I}(\omega)$, $P_{1R}(\omega)$, $P_{1I}(\omega)$, $P_{2R}(\omega)$, $P_{2I}(\omega)$, $P_{0R}(\omega)$ and $P_{0I}(\omega)$ are real-coefficient polynomials. Also let

$$a(\omega) = P_{12R}(\omega) + P_{0R}(\omega), \quad b(\omega) = P_{12I}(\omega) - P_{0I}(\omega), \tag{7}$$

$$c(\omega) = P_{12I}(\omega) + P_{0I}(\omega), \quad d(\omega) = P_{0R}(\omega) - P_{12R}(\omega), \tag{8}$$

$$e(\omega) = P_{1R}(\omega) + P_{2R}(\omega), \quad f(\omega) = P_{1I}(\omega) - P_{2I}(\omega), \tag{9}$$

$$g(\omega) = P_{1I}(\omega) + P_{2I}(\omega), \quad h(\omega) = -P_{1R}(\omega) + P_{2R}(\omega). \tag{10}$$

Theorem 3.1. *The linear retarded dynamical systems with characteristic function* (5) *is delay-independently stable if and only if*: (i) *function* $D(\tau, 0, 0) = P_{12}(\lambda) + P_1(\lambda) + P_2(\lambda) + P_0(\lambda)$ *is Hurwitz stable, and* (ii) *equation* $A_L(\omega) = 0$ *has no non-zero real root* ω, *where*

$$\begin{aligned} A_L(\omega) = {} & [h(\omega)a(\omega) - f(\omega)c(\omega)]^2 + [e(\omega)c(\omega) - g(\omega)a(\omega)]^2 \\ & + [h(\omega)b(\omega) - f(\omega)d(\omega)]^2 + [e(\omega)d(\omega) - g(\omega)b(\omega)]^2 \\ & - [a(\omega)d(\omega) - b(\omega)c(\omega)]^2 - [e(\omega)h(\omega) - f(\omega)g(\omega)]^2. \end{aligned} \tag{11}$$

Proof. Note that $D(\mathrm{i}\omega, \tau_1, \tau_2) = 0$ is equivalent to $\mathrm{e}^{\mathrm{i}\omega(\tau_1+\tau_2)/2} D(\mathrm{i}\omega, \tau_1, \tau_2) = 0$. Based on Lemma 3.1, the key is to prove that condition (ii) in Lemma 3.1 is equivalent to (ii) in Theorem 3.1. The former is true if and only if the equation

$$\begin{bmatrix} p(\omega, \tau_2) & q(\omega, \tau_2) \\ r(\omega, \tau_2) & s(\omega, \tau_2) \end{bmatrix} \begin{bmatrix} \cos \frac{\omega\tau_1}{2} \\ \sin \frac{\omega\tau_1}{2} \end{bmatrix} = 0 \tag{12}$$

has no non-zero real root ω for any given delays τ_1 and τ_2. This in turn equals that, $\forall \omega \in \mathbb{R} \backslash \{0\} =^{\mathrm{def}} \mathbb{R}^*$, the determinant of the coefficient matrix of Eq. (12) is zero for any given τ_2 (Note that τ_1 and τ_2 are independent.), i.e., the equation

$$p(\omega, \tau_2) s(\omega, \tau_2) - q(\omega, \tau_2) r(\omega, \tau_2) = 0 \tag{13}$$

has no non-zero real root ω for any given τ_2, where

$$\begin{aligned} p(\omega, \tau_2) &= [a(\omega) + e(\omega)] \cos \frac{\omega\tau_2}{2} + [b(\omega) - f(\omega)] \sin \frac{\omega\tau_2}{2}, \\ q(\omega, \tau_2) &= [b(\omega) + f(\omega)] \cos \frac{\omega\tau_2}{2} + [e(\omega) - a(\omega)] \sin \frac{\omega\tau_2}{2}, \\ r(\omega, \tau_2) &= [c(\omega) + g(\omega)] \cos \frac{\omega\tau_2}{2} + [d(\omega) - h(\omega)] \sin \frac{\omega\tau_2}{2}, \\ s(\omega, \tau_2) &= [d(\omega) + h(\omega)] \cos \frac{\omega\tau_2}{2} + [g(\omega) - c(\omega)] \sin \frac{\omega\tau_2}{2}. \end{aligned}$$

Note that Eq. (13) can be simplified as

$$\frac{m_2(\omega) + m_0(\omega)}{2} + \frac{-m_2(\omega) + m_1(\omega)}{2} \cos \omega\tau_2 + \frac{m_1(\omega)}{2} \sin \omega\tau_2 = 0,$$

where

$$\begin{aligned} m_2(\omega) &= [b(\omega) - f(\omega)][g(\omega) - c(\omega)] - [d(\omega) - h(\omega)][e(\omega) - a(\omega)], \\ m_1(\omega) &= 2[-f(\omega)d(\omega) + b(\omega)h(\omega) - e(\omega)c(\omega) + a(\omega)g(\omega)], \\ m_0(\omega) &= [a(\omega) + e(\omega)][d(\omega) + h(\omega)] - [b(\omega) + f(\omega)][c(\omega) + g(\omega)]. \end{aligned}$$

So condition (ii) in Theorem 3.1 is equivalent to

$$\left[\frac{m_2(\omega) + m_0(\omega)}{2}\right]^2 > \left[\frac{-m_2(\omega) + m_1(\omega)}{2}\right]^2 + \left[\frac{m_1(\omega)}{2}\right]^2, \quad \forall\omega \in \mathbb{R}^*,$$

i.e., $\forall\omega \in \mathbb{R}^*$, $A_L(\omega) < 0$, which is equivalent to that the equation $A_L(\omega) = 0$ has no non-zero real root ω because $A_L(\omega)$ is a real-coefficient polynomial with leading coefficient being -1 and is even with respect to ω. $\square$

Remark 1. The preposition that the equation $A_L(\omega) = 0$ has no non-zero real root ω (i.e., $\forall\omega \in \mathbb{R}^*, A_L(\omega) \neq 0$), is equivalent to the statement that either (a) or (b) holds true:

(a) This equation has no real root ω;
(b) If this equation has a real root ω, it must be zero.

Based on Theorem 3.1 and assuming that $P_{12}(\lambda) = 0$ in Eq. (5), there exists

Corollary 3.1. *The linear retarded dynamical systems with characteristic function* (4) *is delay-independently stable if and only if*: (i) *function* $D(\lambda, 0, 0) = P_1(\lambda) + P_2(\lambda) + P_0(\lambda)$ *is Hurwitz stable, and* (ii) *equation* $R_L(\omega) = 0$ *has no non-zero real root* ω, *where*

$$\begin{aligned} R_L(\omega) = &-\{P_{0R}^2(\omega) + P_{0I}^2(\omega) - P_{1R}^2(\omega) - P_{1I}^2(\omega) - P_{2R}^2(\omega) - P_{2I}^2(\omega)\}^2 \\ &+ 4[P_{1R}^2(\omega) + P_{1I}^2(\omega)][P_{2R}^2(\omega) + P_{2I}^2(\omega)]. \end{aligned}$$

Let $P_1(\lambda) = P_2(\lambda) = 0$ in Eq. (5) and define $(\tau_1 + \tau_2)$ as τ; there exists

Corollary 3.2. *The linear retarded dynamical systems with the characteristic function*

$$D(\lambda, \tau) = P_{12}(\lambda)e^{-\lambda\tau} + P_0(\lambda)$$

is delay-independently stable if and only if: (i) *function* $D(\lambda, 0) = P_{12}(\lambda) + P_0(\lambda)$ *is Hurwitz stable, and* (ii) *equation* $S_L(\omega) = P_{12R}^2(\omega) + P_{12I}^2(\omega) - P_{0R}^2(\omega) - P_{0I}^2(\omega) = 0$ *has no non-zero real root* ω.

3.2. The case for two dependent delays

Consider the case when $\tau_1 = h_1\tau$ and $\tau_2 = h_2\tau$.

Lemma 3.2. *The linear retarded dynamical system with characteristic function* (5) *is delay-independently stable if and only if*: (i) *function* $D(\lambda, 0) = P_{12}(\lambda) + P_1(\lambda) + P_2(\lambda) + P_0(\lambda)$ *is Hurwitz stable, and* (ii) *equation* $D(\mathrm{i}\omega, \tau) = 0$ *has no non-zero real root* ω *for any given* τ.

Proof. The proof is similar to Lemma 3.1. □

Theorem 3.2. *The linear retarded dynamical systems with characteristic function* (5) *is delay-independently stable if and only if*: (i) *function* $D(\lambda, 0) = P_{12}(\lambda) + P_1(\lambda) + P_2(\lambda) + P_0(\lambda)$ *is Hurwitz stable, and* (ii) $\forall \omega \in \mathbb{R}^*$, *two equations* $f_1(u) = 0$ *and* $f_2(u) = 0$ *have no common real root u, where* $f_1(u), f_2(u)$ *are certain functions whose expressions are determined in the following proof.*

Proof. Note that condition (i) in Lemma 3.2 is the same as (i) in Theorem 3.2. Based on Lemma 3.2, the key is to prove that condition (ii) in Lemma 3.2 is equivalent to (ii) in Theorem 3.2 when the same condition (i) is held true. Note that condition (ii) in Lemma 3.2 is equivalent to that $\forall \omega \in \mathbb{R}^*$, equations $C_L(\omega, \tau) = 0$ and $D_L(\omega, \tau) = 0$ have no common root τ, where

$$\begin{aligned} C_L(\omega, \tau) &= p(\omega, h_2\tau) \cos \frac{\omega h_1 \tau}{2} + q(\omega, h_2\tau) \sin \frac{\omega h_1 \tau}{2}, \\ D_L(\omega, \tau) &= r(\omega, h_2\tau) \cos \frac{\omega h_1 \tau}{2} + s(\omega, h_2\tau) \sin \frac{\omega h_1 \tau}{2} \end{aligned} \tag{14}$$

in which

$$p(\omega, h_2\tau) = [a(\omega) + e(\omega)] \cos \frac{\omega h_2 \tau}{2} + [b(\omega) - f(\omega)] \sin \frac{\omega h_2 \tau}{2}, \tag{15}$$

$$q(\omega, h_2\tau) = [b(\omega) + f(\omega)] \cos \frac{\omega h_2 \tau}{2} + [e(\omega) - a(\omega)] \sin \frac{\omega h_2 \tau}{2}, \tag{16}$$

$$r(\omega, h_2\tau) = [c(\omega) + g(\omega)] \cos \frac{\omega h_2 \tau}{2} + [d(\omega) - h(\omega)] \sin \frac{\omega h_2 \tau}{2}, \tag{17}$$

$$s(\omega, h_2\tau) = [d(\omega) + h(\omega)] \cos \frac{\omega h_2 \tau}{2} + [g(\omega) - c(\omega)] \sin \frac{\omega h_2 \tau}{2}, \tag{18}$$

where $a(\omega) \sim h(\omega)$ are calculated by formulae (7)–(10). Let $\cos \omega\tau/2 = x$, $\sin \omega\tau/2 = y$. Then it is easy to show that

$$\cos \frac{\omega h_i \tau}{2} = \sum_{j=0}^{h_i} \alpha_{ij} x^{h_i - j} y^j, \quad \sin \frac{\omega h_i \tau}{2} = \sum_{j=0}^{h_i} \beta_{ij} x^{h_i - j} y^j, \tag{19}$$

where $i = 1, 2$; α_{ij}, β_{ij} are certain constants (note that $\alpha_{i0} \neq 0$, $\beta_{i0} = 0$). So formulae (15)–(18) can be rewritten as

$$\begin{aligned} p(\omega, h_2\tau) &= \sum_{j=0}^{h_2} \gamma_{1j}(\omega) x^{h_2 - j} y^j, \quad q(\omega, h_2\tau) = \sum_{j=0}^{h_2} \gamma_{2j}(\omega) x^{h_2 - j} y^j, \\ r(\omega, h_2\tau) &= \sum_{j=0}^{h_2} \gamma_{3j}(\omega) x^{h_2 - j} y^j, \quad s(\omega, h_2\tau) = \sum_{j=0}^{h_2} \gamma_{4j}(\omega) x^{h_2 - j} y^j, \end{aligned} \tag{20}$$

where $\gamma_{ij}(\omega)$ is a certain function (for $i = 1, 2, 3, 4$; $j = 0, 1, \ldots, h_2$); especially there exists

$$\gamma_{10}(\omega) = [a(\omega) + e(\omega)]\alpha_{20}, \quad \gamma_{20}(\omega) = [b(\omega) + f(\omega)]\alpha_{20},$$
$$\gamma_{30}(\omega) = [c(\omega) + g(\omega)]\alpha_{20}, \quad \gamma_{20}(\omega) = [d(\omega) + h(\omega)]\alpha_{20}.$$

Then Eq. (14) can be transformed into

$$C_L(\omega, \tau) = \sum_{j=0}^{h_1+h_2} \xi_j(\omega) x^{h_1+h_2-j} y^j,$$
$$D_L(\omega, \tau) = \sum_{j=0}^{h_1+h_2} \eta_j(\omega) x^{h_1+h_2-j} y^j$$

by using Eqs. (19) and (20), where

$$\xi_0(\omega) = [a(\omega) + e(\omega)]\alpha_{20}\alpha_{10}, \quad \eta_0(\omega) = [b(\omega) + f(\omega)]\alpha_{20}\alpha_{10}.$$

Note that $D(\lambda, 0)$ is Hurwitz stable, so there exists

$$\xi_0(\omega) + \mathrm{i}\eta_0(\omega) = D(\mathrm{i}\omega, 0) \neq 0, \quad \forall \omega \in \mathbb{R}^*.$$

This means $\forall \omega \in \mathbb{R}^*$, $\xi_0(\omega)$ and $\eta_0(\omega)$ do not equal to zero simultaneously. Let $\frac{x}{y} = u$, then condition (ii) in Lemma 3.2 is equivalent to $\forall \omega \in \mathbb{R}^*$, equations $f_1(u) = 0$ and $f_2(u) = 0$ have no common real root u, where

$$f_1(u) = \sum_{j=0}^{h_1+h_2} \xi_j(\omega) u^{h_1+h_2-j}, \quad f_2(u) = \sum_{j=0}^{h_1+h_2} \eta_j(\omega) u^{h_1+h_2-j}.$$

Thus Theorem 3.2 is proved. □

Remark 2. The equivalence of two real-coefficient polynomials $s_1(x) = \sum_{i=0}^{n} a_i x^{n-i}$ and $s_2(x) = \sum_{i=0}^{n} b_i x^{n-i}$ ($a_i, b_i \in \mathbb{R}$; $n \geqslant 1$; $a_0^2 + b_0^2 \neq 0$) have no common real root x is either (1) $s_1(x)$ and $s_2(x)$ have no common root x, or (2) any common roots of $s_1(x)$ and $s_2(x)$ are not real. Proposition (1) can be further simplified as $res[s_1(x), s_2(x)] = \det[R(s_1, s_2)] \neq 0$, where $res[s_1(x), s_2(x)]$ is the resultant of $s_1(x)$ and $s_2(x)$, and $R(s_1, s_2)$ is the corresponding Sylvester resultant matrix [14, pp. 408–426] (with the size $2n \times 2n$), where

$$R(s_1, s_2) = \begin{bmatrix} a_0 & a_1 & \cdots & \cdots & a_n & 0 & \cdots & 0 \\ 0 & a_0 & a_1 & \cdots & \cdots & a_n & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & & & \ddots & \vdots \\ 0 & \cdots & 0 & a_0 & a_1 & \cdots & \cdots & a_n \\ b_0 & b_1 & \cdots & \cdots & b_n & 0 & \cdots & 0 \\ 0 & b_0 & b_1 & \cdots & \cdots & b_n & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & & & \ddots & \vdots \\ 0 & \vdots & 0 & b_0 & b_1 & \cdots & \cdots & b_n \end{bmatrix}.$$

Proposition (2), generally speaking, cannot be easily simplified. However, for the case $h_1 = h_2 = 1$ (accordingly $\tau_1 = \tau_2 = \tau$), it can be simplified as det $[R(s_1, s_2)] = 0$ but either $s_1(x)$ or $s_2(x)$ has no real root x.

Specifically, let $\tau_1 = \tau_2 = \tau$ in Eq. (5) and $P_{1+2}(\lambda) = P_1(\lambda) + P_2(\lambda)$. Based on Theorem 3.2 and Remark 2, there exists

Corollary 3.3. *The linear retarded dynamical systems with the characteristic function*

$$D(\lambda, \tau) = P_{12}(\lambda)\mathrm{e}^{-2\lambda\tau} + P_{1+2}(\lambda)\mathrm{e}^{-\lambda\tau} + P_0(\lambda) \tag{21}$$

is delay-independently stable if and only if: (i) *function* $D(\lambda, 0) = P_{12}(\lambda) + P_{1+2}(\lambda) + P_0(\lambda)$ *is Hurwitz stable, and* (ii) $\forall\omega\in\mathbb{R}^*$, (1) $E_L(\omega)\neq 0$ *or* (2) $E_L(\omega) = 0$ *but either* $F_L(\omega)<0$ *or* $G_L(\omega)<0$, *where*

$$\begin{aligned} E_L(\omega) = &\{P_{1+2R}(\omega)[P_{0R}(\omega) - P_{12R}(\omega)] - P_{1+2I}(\omega)[P_{12I}(\omega) - P_{0I}(\omega)]\}^2 \\ &+ \{P_{1+2R}(\omega)[P_{0I}(\omega) + P_{12I}(\omega)] - P_{1+2I}(\omega)[P_{0R}(\omega) + P_{12R}(\omega)]\}^2 \\ &+ [P_{0R}^2(\omega) + P_{0I}^2(\omega) - P_{12R}^2(\omega) - P_{12I}^2(\omega)]^2, \end{aligned}$$

$$\begin{aligned} F_L(\omega) &= [P_{12R}(\omega) + P_{0R}(\omega)]^2 + [P_{12I}(\omega) - P_{0I}(\omega)]^2 - P_{1+2R}^2(\omega), \\ G_L(\omega) &= [P_{12I}(\omega) + P_{0I}(\omega)]^2 + [P_{0R}(\omega) - P_{12R}(\omega)]^2 - P_{1+2I}^2(\omega). \end{aligned}$$

Proof. It is obvious that, $\forall\omega\in\mathbb{R}^*$, both $f_1(u)$ and $f_2(u)$ are polynomials whose degrees are not more than two since $h_1 = h_2 = 1$, where

$$\begin{aligned} \xi_0(\omega) &= a(\omega) + e(\omega), \quad \xi_1(\omega) = 2b(\omega), \quad \xi_2(\omega) = e(\omega) - a(\omega), \\ \eta_0(\omega) &= c(\omega) + g(\omega), \quad \eta_1(\omega) = 2d(\omega), \quad \eta_2(\omega) = g(\omega) - c(\omega), \end{aligned}$$

in which $\xi_0(\omega)$ and $\eta_0(\omega)$ do not equal to zero simultaneously. By Theorem 3.2 and Remark 2, condition (ii) in Theorem 3.2 is equivalent to that, $\forall\omega\in\mathbb{R}^*$, (1) det$[R(f_1,f_2)]\neq 0$, or, (2) det$[R(f_1,f_2)] = 0$ but either $f_1(u) = 0$ or $f_2(u) = 0$ has no real root u. The equivalence of (1) is that det$[R(f_1,f_2)] = 4H_L(\omega)\neq 0$, where

$$H_L(\omega) = [e(\omega)d(\omega) - g(\omega)b(\omega)]^2 + [e(\omega)c(\omega) - g(\omega)a(\omega)]^2 - [a(\omega)d(\omega) - b(\omega)c(\omega)]^2 = E_L(\omega).$$

The equivalence of Eq. (2) is that $E_L(\omega) = 0$ but either $discr[f_1(u)] = 4[a^2(\omega) + b^2(\omega) - e^2(\omega)] = 4F_L(\omega)<0$ or $discr[f_2(u)] = 4[c^2(\omega) + d^2(\omega) - g^2(\omega)] = 4G_L(\omega)<0$, where $discr[f_1(u)]$ and $discr[f_2(u)]$ are discriminants of $f_1(u) = 0$ and $f_2(u) = 0$, respectively. Thus Corollary 3.3 is proved. □

Remark 3. Corollary 3.3 can be further simplified under certain conditions. For example, suppose $\forall\omega\in\mathbb{R}^*$, $a(\omega)d(\omega) - b(\omega)c(\omega) \overset{\text{def}}{=} I_L(\omega)\neq 0$, then condition (ii) in Corollary 3.3 is equivalent to $\forall\omega\in\mathbb{R}^*$, $E_L(\omega)\neq 0$. This is because sub-condition (2) in (ii) is not held true for any $\omega\in\mathbb{R}^*$, which is proved as follows. Suppose for a given $\theta\in\mathbb{R}^*$, $E_L(\theta) = 0$, i.e.,

$$\begin{aligned} E_L(\theta) = &[c^2(\theta) + d^2(\theta)]e^2(\theta) - 2g(\theta)[a(\theta)c(\theta) + b(\theta)d(\theta)]e(\theta) \\ &+ g^2(\theta)[a^2(\theta) + b^2(\theta)] - [a(\theta)d(\theta) - b(\theta)c(\theta)]^2 = 0. \end{aligned} \tag{22}$$

This means the discriminant of Eq. (22) ($e(\theta)$ acts as an unknown number) is non-negative, i.e.,

$$discr[E_L(\theta)] = 4I_L^2(\theta)[c^2(\theta) + d^2(\theta) - g^2(\theta)] \geqslant 0.$$

Recall that $\forall \omega \in \mathbb{R}^*$, $I_L(\omega) \neq 0$, then $c^2(\theta) + d^2(\theta) \geqslant 0$. Similarly, $a^2(\theta) + b^2(\theta) \geqslant 0$. Note that θ is arbitrary, so $\forall \omega \in \mathbb{R}^*$, $a^2(\omega) + b^2(\omega) \geqslant 0$ and $c^2(\omega) + d^2(\omega) \geqslant 0$ when $\forall \omega \in \mathbb{R}^*$, $E_L(\omega) = 0$. This shows sub-condition (2) is not true for any $\omega \in \mathbb{R}^*$.

4. Application

In this section several examples are given to illustrate how to get the delay-independent stability conditions. Two kinds of systems are considered: all-parameter systems or part-parameter ones. Here 'all-parameter' means that each element of A_1, A_2 and A_3 in Eq. (1) is unknown, whereas 'part-parameter' means that some elements of A_1, A_2 and A_3 in (1) are unknown.

4.1. All-parameter systems

Before the analyses, a fact should be pointed out that if the system matrix A_i and the state vectors $x(t)$, $x(t-\tau_i)$ $(i = 1, 2, \ldots, N)$ in Ref. [1] are considered in complex spaces $\mathbb{C}^{n\times n}$ and $\mathbb{C}^{n\times 1}$, respectively, the relevant delay-independent stability results, such as Theorem 2.3 in Ref. [1], remain right. This fact is also applicable to those in this paper.

Consider the linear retarded system

$$\dot{x}(t) = Ax(t) + r_1 Ax(t-\tau_1) + r_2 Ax(t-\tau_2), \tag{23}$$

where $A \in \mathbb{R}^{n\times n}$ is a stable matrix; τ_1 and τ_2 are independent; $r_1, r_2 \in \mathbb{R}$. Suppose the eigenvalue sequence of A is $\{\mu_i | \mathrm{Re}(\mu_i) < 0,\ i = 1, 2, \ldots, n\}$, then the characteristic function of Eq. (23) is

$$D(\lambda, \tau_1, \tau_2) = \left| \lambda I - A\left(1 + \sum_{k=1}^{2} r_k \mathrm{e}^{-\lambda\tau_k}\right)\right| = \prod_{i=1}^{n}\left[\lambda - \left(1 + \sum_{k=1}^{2} r_k \mathrm{e}^{-\lambda\tau_k}\right)\mu_i\right].$$

So system (23) is delay-independent stable if and only if each function $D_i(\lambda, \tau_1, \tau_2) = \lambda - (1 + \sum_{k=1}^{2} r_k \mathrm{e}^{-\lambda\tau_k})\mu_i$ is stable, which is equivalent to (i) $\mathrm{Re}[\mu_i(1 + r_1 + r_2)] < 0$, and (ii) $(|r_1| + |r_2|)|\mu_i| < |\mathrm{Re}(\mu_i)|$ if $\mathrm{Im}(\mu_i) \neq 0$; $(|r_1| + |r_2|)|\mu_i| \leqslant |\mathrm{Re}(\mu_i)|$ if $\mathrm{Im}(\mu_i) = 0$, $i = 1, 2, \ldots, n$. The conditions can be reduced either to (1) $-1 < r_1 + r_2 \leqslant |r_1| + |r_2| \leqslant 1$, if each μ_i satisfies $\mathrm{Im}(\mu_i) = 0$; or (2) $-1 < r_1 + r_2 \leqslant |r_1| + |r_2| < \min_{1\leqslant i\leqslant n} |\mathrm{Re}(\mu_i)|/|\mu_i|$, if there exist some μ_i such that $\mathrm{Im}(\mu_i) \neq 0$. Now consider a special case: $r_1 = r$, $r_2 = 0$, $\tau_1 = \tau$, then conditions (1) and (2) can be reduced to (1′) $r \in (-1, 1]$ and (2′) $r \in (-\min_{1\leqslant i\leqslant n} |\mathrm{Re}(\mu_i)|/|\mu_i|,\ \min_{1\leqslant i\leqslant n} |\mathrm{Re}(\mu_i)|/|\mu_i|)$, respectively, which are identical to that in Ref. [1].

Consider the linear retarded system with the characteristic function

$$D(\lambda, \tau) = \lambda - a - b\mathrm{e}^{-\lambda\tau_1} - c\mathrm{e}^{-\lambda\tau_2},$$

where $a, b, c \in \mathbb{C}$; τ_1 and τ_2 are independent. By Corollary 3.1, the function is stable if and only if: (1) $\mathrm{Re}(a + b + c) < 0$ and $|b| + |c| \leqslant |a|$, if $\mathrm{Im}(a) = 0$; (2) $\mathrm{Re}(a + b + c) < 0$ and $|b| + |c| < |\mathrm{Re}(a)|$, if $\mathrm{Im}(a) \neq 0$. This result provides the delay-independent stability analysis for the systems in [6,15–17].

4.2. Part-parameter systems

Consider a stirred tank model [18] in the form

$$\dot{x}(t) = \begin{bmatrix} -\dfrac{1}{2\theta} & 0 \\ 0 & -\dfrac{1}{\theta} \end{bmatrix} x(t) + \begin{bmatrix} 1 & 1 \\ \dfrac{c_1 - c_0}{V_0} & \dfrac{c_2 - c_0}{V_0} \end{bmatrix} u(t), \tag{24}$$

where $x(t) = [x_1(t)\ x_2(t)]^{\mathrm{T}} \in \mathbb{R}^{2\times 1}$ is the state vector and $u(t) = [u_1(t)\ u_2(t)]^{\mathrm{T}} \in \mathbb{R}^{2\times 1}$ is the control vector; $c_1 = k_1$ and $c_2 = k_2$ are positive parameters; $u_1(t)$ and $u_2(t)$ are defined by

$$u_1(t) = \sum_{j=1}^{2} g_{1j} x_j(t - \tau_1),$$

$$u_2(t) = \sum_{j=1}^{2} g_{2j} x_j(t - \tau_2),$$

where $g_{ij} \in \mathbb{R}$ are constant feedback gains, τ_1 and τ_2 are time delays. For physical meaning of $\theta, V_0, c_0, c_1, c_2$ one can refer to [18]. The delay-independent stability of Eq. (24) is analyzed as follows with the given constants: $\theta = 10$, $V_0 = 1$, $c_0 = 1.25$, $g_{11} = -0.01875$, $g_{12} = 0.05$, $g_{21} = -0.00625$, $g_{22} = -0.05$ and the limited condition $|k_2 - k_1| < 4$.

4.2.1. The case of two independent delays

The characteristic function is

$$\begin{aligned} D(\lambda, \tau_1, \tau_2) = {} & \lambda^2 + 0.15\lambda + 0.005 + 0.00125(k_2 - k_1)\mathrm{e}^{-\lambda(\tau_1+\tau_2)} \\ & + [(0.08125 - 0.05k_1)\lambda + 0.005 - 0.0025k_1]\mathrm{e}^{-\lambda\tau_1} \\ & + [(-0.05625 + 0.05k_2)\lambda - 0.0025 + 0.0025k_2]\mathrm{e}^{-\lambda\tau_2}. \end{aligned}$$

System Eq. (24) is delay-independently stable if and only if: (i) function $D(\lambda, 0, 0)$ is Hurwitz stable, i.e.,

$$0 < 0.175 - 0.05k_1 + 0.05k_2, \tag{25}$$

$$0 < 0.0075 - 0.00375k_1 + 0.00375k_2, \tag{26}$$

and (ii) the equation $\bar{A}_L(\omega) = -A_L(\omega) = 0$ has no non-zero real root ω, where

$$\bar{A}_L(\omega) = \omega^8 + b_1\omega^6 + b_2\omega^4 + b_3\omega^2 + b_4, \tag{27}$$

where b_1, b_2, b_3 and b_4 are functions of k_1 and k_2. Here b_i can be easily calculated according to Theorem 3.1. The expressions of b_i are omitted because they are too complicated.

For condition (ii), two main problems need to be solved. One is to assure that Eq. (27) has no real root ω, which can be solved by the generalized Sturm theory [12,13,18] or by directly applying the relevant results in Section 4.3.1 in Ref. [9].

The other is to assure that the only real roots of polynomial (27) are zeros. There are two cases: (1) $b_4 = 0$ and the equation $\omega^6 + b_1\omega^4 + b_2\omega^2 + b_3 = 0$ has no real root ω, (2) $b_4 = b_3 = 0$ and the equation $\omega^4 + b_1\omega^2 + b_2 = 0$ has no non-zero real root ω. Case (1) can surely be solved by the

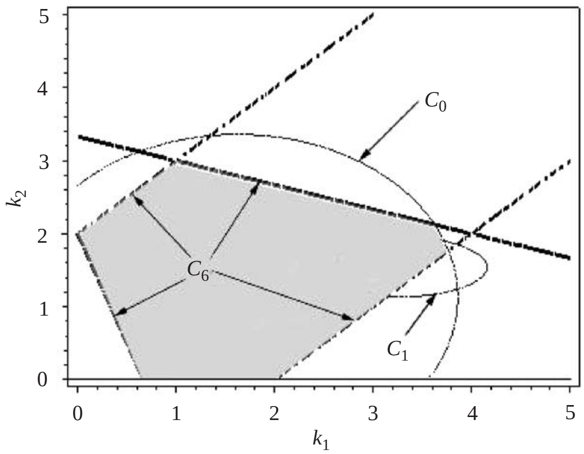

Fig. 1. The delay-independently stable region of the retarded stirred tank model with τ_1 and τ_2 independent and with $\theta = 10$, $V_0 = 1$, $c_0 = 1.25$, $g_{11} = -0.01875$, $g_{12} = 0.05$, $g_{21} = -0.00625$, $g_{22} = -0.05$. The curves are given by $C_i : d_i = 0$, $i = 0, 1, 6$.

generalized Sturm theory, but it is reduced to the condition $b_4 = 0$, $b_3 > 0$ when the inequality $b_3 > 0$ contains $b_1 \geqslant 0, b_2 \geqslant 0$. Case (2) is equivalent to either $b_4 = b_3 = 0$, $4b_2 > b_1^2$ or $b_4 = b_3 = 0$, $b_2 \geqslant 0$, $b_1 \geqslant 0$.

Condition (i) results in the region named $R_0 : \{(k_1, k_2)|k_1 - k_2 - 2.0 < 0\}$, whereas condition (ii) results in the region named R_1 which contains two parts: one is determined by plotting the curves $d_0 = 0$, $d_1 = 0, \ldots, d_6 = 0$ and finding the intersected areas according to the sign tables for polynomial (27), where $d_0 \sim d_6$ are calculated according to formula (48) in Ref. [9] by substituting b_i in Eq. (27). The other consists of four segments: $s_1 = \{k_2 = k_1 + 2, 0.143 \leqslant k_1 \leqslant 0.982\}$, $s_2 = \{k_2 = 2 - 3k_1, 0.019 < k_1 \leqslant 2/3\}$, $s_3 = \{k_2 = k_1 - 2, 2 < k_1 \leqslant 3.570\}$, $s_4 = \{k_2 = 3 - 3k_1, 1.011 \leqslant k_1 \leqslant 2.208\}$.

The delay-independent stable region, $R_0 \cap R_1$, is the shaded part in Fig. 1. [2] This region consists of the segments s_1, s_2, s_4 (these segments are on the curve C_6) and the inner circumscribed by the curves C_0 and C_6.

4.2.2. The case of two dependent delays

Only consider the case when $\tau_1 = \tau_2 = \tau$. The characteristic function is

$$D(\lambda, \tau_1, \tau_2) = \lambda^2 + 0.15\lambda + 0.005 + 0.00125(k_2 - k_1)e^{-2\lambda\tau} + 0.0025[(10 - 20k_1 + 20k_2)\lambda + 1 - k_1 + k_2]e^{-\lambda\tau}.$$

Note that $\forall \omega \in \mathbb{R}^*$, $I_L(\omega) = \omega^4 + 0.0125\omega^2 + 1.5625 \times 10^{-6}[16 - (k_2 - k_1)^2] > 0$ by recalling the limited condition $|k_2 - k_1| < 4$. Based on Corollary 3.3 and Remark 3, the system is delay-independently stable if and only if: (i) function $D(\lambda, 0)$ is Hurwitz stable, which is the same as Eqs. (25) and (26), and (ii) $\forall \omega \in \mathbb{R}^*$, $E_L(\omega) \neq 0$.

[2] Note in Fig. 1, some curves or their branches are not shown for the clarity of figures; they do not affect the results.

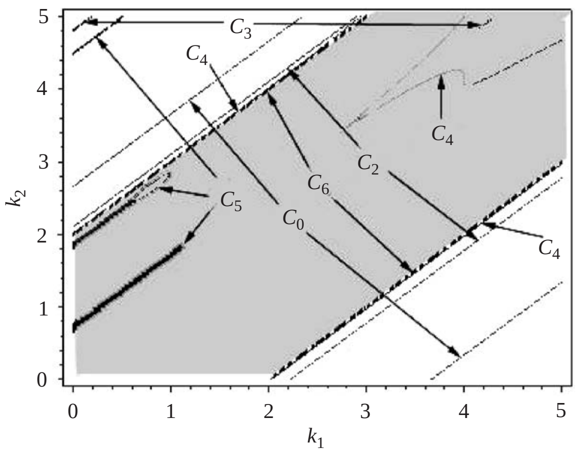

Fig. 2. The delay-independently stable region of the stirred tank model with $\tau_1 = \tau_2$ and with $\theta = 10$, $V_0 = 1$, $c_0 = 1.25$, $g_{11} = -0.01875$, $g_{12} = 0.05$, $g_{21} = -0.00625$, $g_{22} = -0.05$. The curves are given by $C_i{:}d_i = 0,\ i = 0, 2, 3, 4, 5, 6$.

The region R_0 for condition (i) is $\{(k_1, k_2) \mid k_1 - k_2 - 2.0 < 0\}$. The region R_1 for condition (ii) contains two parts: one is determined by plotting the curves $d_0 = 0, d_1 = 0, \ldots, d_6 = 0$ and finding the intersected areas according to the sign tables of Eq. (27), where $d_0 \sim d_6$ are calculated according to formula (48) in Ref. [9] with substituting of b_i in Eq. (27). The other consists of two rays: $r_1 = \{k_2 = k_1 + 2, k_1 > 0\}$, $r_2 = \{k_2 = k_1 - 2, k_1 > 2\}$. The delay-independent stable region, $R_0 \cap R_1$, is the shaded part in Fig. 2. [3] This region consists of the ray r_1 (r_1 coincides with the curve C_6) and the inner circumscribed by C_6 and the k_1, k_2 axes. This region is much larger than that for independent delays in Fig. 1.

5. Conclusion

This paper develops sufficient and necessary delay-independent stability criteria for a class of retarded dynamical systems with two discrete time delays. The criteria can be used to analyze the delay-independent stability of practical linear retarded systems. It is also shown that the stability criterion for systems with two dependent delays is more complicated than that for the systems with two independent delays because the former needs to check the non-existence of common real roots of two polynomials.

In the application of the criteria, several examples are given which either are identical to or improved upon the corresponding results in the pioneers' works. And the systems considered are limited in the linear retarded systems in the form of Eq. (1) with parameters. As for those without parameters, the criteria in this paper can also work well because the key of the delay-independent stability problem is that some polynomials with given coefficients have no non-zero roots or no non-zero common roots, which is still in the solvable field of the generalized Sturm theory when necessary.

[3] Note in Fig. 2, some curves or their branches are not shown for the clarity of figures; they do not affect the results.

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